2.7 D Theorem: If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, If $\lim_{n\to\infty} s_n = L$ and if $\lim_{n\to\infty} t_n = M$

then $\lim_{n\to\infty} (s_n - t_n) = L - M$.

Proof: Given $\lim_{n\to\infty} s_n = L$ —— (1)

Also given $\lim_{n\to\infty} t_n = M$, with $c = -1$

$\lim_{n\to\infty} c t_n = cM$ $\therefore \lim_{n\to\infty} (-t_n) = (-M)$ —— (2)

$\therefore \lim_{n\to\infty} (s_n + (-t_n)) = L + (-M)$

$\lim_{n\to\infty} (s_n - t_n) = L - M$.

2.7E Corollary: If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences of real numbers, if $s_n \leq t_n$ $(n \in I)$ and if $\lim_{n\to\infty} s_n = L$, $\lim_{n\to\infty} t_n = M$, then $L \leq M$.

Proof

$\lim_{n\to\infty} (t_n - s_n) = M - L$

given $t_n \geq s_n \Rightarrow (t_n - s_n) \geq 0$ $\forall n \in I$

$\therefore$ sequence $\{(t_n - s_n)\}_{n=1}^{\infty}$ sequence of non-negative numbers.

$M - L \geq 0$

$\therefore$ $M \geq L$

2.7F LEMMA: If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers, which converges to L, then $\{s_n^2\}_{n=1}^{\infty}$ converges to L^2.

We must prove $\lim s_n^2 = L^2$

That is to prove, given $\epsilon_1 > 0$ there exists $N \in I$ (4)
Such that $|s_n^2 - L^2| < \epsilon_1 \quad \forall\, n \geq N$.

Given sequence $\{s_n\}_{n=1}^{\infty}$ is convergent.

$\therefore$ It is bounded sequence. Thus for some $M > 0$

$$|s_n| \leq M \qquad \forall\, n \in I \quad \text{—— (1)}$$

$$\therefore |s_n + L| \leq |s_n| + |L|$$

$$|s_n + L| \leq M + |L| \qquad \forall\, n \in I \quad \text{—— (2)}$$

$\lim\limits_{n\to\infty} s_n = L \;\therefore$ By defn, given $\epsilon_1 > 0$ there exists $N \in I$ such that

$$|s_n - L| < \frac{\epsilon_1}{M + |L|} \qquad \forall\, n \geq N \quad \text{—— (3)}$$

$\therefore \forall\, n \geq N$

$$|s_n + L|\,|s_n - L| < (M + |L|) \cdot \frac{\epsilon_1}{M + |L|}$$

$$|s_n^2 - L^2| < \epsilon_1 \qquad \forall\, n \geq N$$

$$\Rightarrow \lim_{n\to\infty} s_n^2 = L^2$$

2.7G Theorem: If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\lim\limits_{n\to\infty} s_n = L$ and $\lim\limits_{n\to\infty} t_n = M$ then $\lim\limits_{n\to\infty} s_n t_n = LM$.

Proof: Given $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} t_n = M$

$\therefore \lim_{n\to\infty} (s_n + t_n) = L+M$

$\Rightarrow \lim_{n\to\infty} (s_n + t_n)^2 = (L+M)^2$ —— (1)

Also $\lim_{n\to\infty} (s_n - t_n) = (L-M)$

$\therefore \lim_{n\to\infty} (s_n - t_n)^2 = (L-M)^2$ —— (2)

$\therefore \lim_{n\to\infty} \left((s_n+t_n)^2 - (s_n-t_n)^2\right) = \left[(L+M)^2 - (L-M)^2\right]$

$\therefore \lim_{n\to\infty} (4 s_n t_n) = 4LM$

$\lim_{n\to\infty} s_n t_n = LM.$

2.7H Lemma: If $\{t_n\}_{n=1}^{\infty}$ is a sequence of real numbers.

If $\lim_{n\to\infty} t_n = M$ where $M \neq 0$, then $\lim_{n\to\infty} \left(\frac{1}{t_n}\right) = \frac{1}{M}$

Proof Either $M>0$ or $M<0$, we shall prove the lemma in the case $M>0$

(The case $M<0$ can be proved by applying the first case to $\{-t_n\}_{n=1}^{\infty}$.)

So we assume $M>0$

To prove $\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{M}$

(i.e) To prove $\left|\frac{1}{t_n} - \frac{1}{M}\right| < \varepsilon \quad \forall n \geq N.$

Given $\lim_{n\to\infty} t_n = M$ By defn given $\varepsilon > 0$ there exists $N_1 \in I$ such that

$$|t_n - M| < \varepsilon \quad \forall n \geq N_1$$

Since $M > 0$

$\frac{M}{2} > 0$

Let $\varepsilon < \frac{M}{2}$.

let $\varepsilon < \frac{M}{2}$

$$\Rightarrow |t_n - M| < \frac{M}{2} \quad \forall n \geq N_1$$

$$-\frac{M}{2} < t_n - M < \frac{M}{2}$$

$$M - \frac{M}{2} < t_n \Rightarrow t_n > \frac{M}{2}$$

$$\Rightarrow \frac{1}{t_n} < \frac{2}{M} \quad \underline{\forall n \geq N_1} \quad (1)$$

$$\Rightarrow \cancel{\frac{1}{t_n^2} < \frac{4}{M^2}} \quad \cancel{\forall n \geq N_1} \text{ ——— } (1)$$

In addition, there exists $N_2 \in I$ such that

$$|t_n - M| < \frac{M^2 \varepsilon}{4} \quad \forall n \geq N_2 \text{ ——— } (2)$$

Let $N = \max\{N_1, N_2\}$

$\therefore \forall n \geq N$, then $\frac{1}{t_n^2} < \frac{4}{M_2}$ and $|t_n - M| < \frac{M^2 \varepsilon}{4}$

$\therefore \forall n \geq N$

now $\left|\frac{1}{t_n} - \frac{1}{M}\right| = \left|\frac{M - t_n}{t_n M}\right| = \left|\frac{t_n - M}{t_n M}\right|$

$$= \frac{|t_n - M|}{|t_n||M|}$$

$\cancel{|t_n - M|}$

$$\left|\frac{1}{t_n} - \frac{1}{M}\right| = \frac{|t_n - M|}{t_n M}$$

$$< \frac{M^2 \epsilon}{2} \cdot \frac{2}{M} \cdot \frac{1}{M}$$

$$\left|\frac{1}{t_n} - \frac{1}{M}\right| < \epsilon \quad \forall n \geq N$$

$\Rightarrow \lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{M}$ hence proved

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2.7 I Theorem: If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\lim_{n\to\infty} s_n = L$ and if $\lim_{n\to\infty} t_n = M$ where $M \neq 0$, then $\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{L}{M}$.

Given $\lim_{n\to\infty} s_n = L$ —— ①

Also given $\lim_{n\to\infty} t_n = M$ and $M \neq 0$

∴ By the previous result (2.7 H Lemma)

$$\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{M} \quad \text{—— ②}$$

using Theorem 2.7 G.

$$\lim_{n\to\infty} s_n \cdot \frac{1}{t_n} = L \cdot \frac{1}{M}$$

$\Rightarrow \lim_{n\to\infty} \frac{s_n}{t_n} = \frac{L}{M}$ where $M \neq 0$
